Closing Wed: HW_6A,6B,6C (7.4,7.5,7.7)
Office Hours Today: 1:30-3:00pm (Com B-006)

7.4 Partial Fractions

Goal: A general method to integrate rational functions. Also, this is an important algebraic method for simplifying fractions which you will use in many, many other math courses.

Motivation:

$$\int \frac{x^3 + 4x - 4}{x^2(x^2+4)} dx \quad = ??$$

We will learn to break up the fraction into:

$$\frac{x^3 + 4x - 4}{x^2(x^2+4)} = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^2+4}$$

Then we integrate each part separately.

Entry Task:

1. Using our table of integrals, integrate

$$\int \frac{x^3 + 4x - 4}{x^2(x^2+4)} dx = \int \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^2+4} dx$$

2. *A fraction skill*: Reduce the "improper" fraction into a simplified mixed fraction:

$$\frac{576}{11} = ? + \frac{?}{11}$$

Partial Fraction Decomposition

Step 1: Is the fraction *reduced*?
reduced - highest power on top smaller than the highest power on bottom.

If yes, move to step 2.
If not, divide, then move to step 2.

Example:

$$\int \frac{x^2 + x}{x + 3}\,dx$$

Partial Fractions Method Summary

Step 1: Divide, if needed (on previous page).

Step 2: Factor Denominator and write out form of decomposition.

i) Distinct Linear:

$$\frac{x^2-3}{x(x-1)(x+4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+4}$$

ii) Repeated Linear:

$$\frac{5+2x}{(x+3)(x-2)^3} = \frac{A}{x+3} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}$$

iii) Irreducible Quadratic:

$$\frac{4x}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$$

Step 3: Solve for A, B, C ….

Step 4: Integrate

Here are examples of the **<u>only types of integrals</u>** you need for this section:

$$\int \frac{1}{2x+5}\,dx = \frac{1}{2}\ln|2x+5| + C$$

$$\int \frac{1}{(x-4)^2}\,dx = -\frac{1}{x-4} + C$$

$$\int \frac{1}{(x+7)^3}\,dx = -\frac{1}{2}\frac{1}{(x+7)^2} + C$$

$$\int \frac{1}{x^2+9}\,dx = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

$$\int \frac{x}{x^2+9}\,dx = \frac{1}{2}\ln|x^2+9| + C$$

As you see, the method rewrites **any** rational function problem to a sum of these types.

Example:

$$\int \frac{x+1}{x^2-4}\,dx$$

Example:

$$\int \frac{x+1}{x^3+3x^2}\,dx$$

Example:

$$\int \frac{x^2 - x + 6}{x^3 + 3x}\,dx$$

Example:

$$\int \frac{x}{x^2 + 4x + 5}\, dx$$

How to integrate

A. Look for simplifications/substitutions

B. Products/Logs/Inverse Trig → BY PARTS
   Sin/Cos/Tan/Sec combos → TRIG
   Quadratic (under a radical) → TRIG SUB
   Rational Function → PART. FRAC.

C. If nothing seems to work, substitution.
   (u = inside, u = $\sqrt{}$, u = trig, u = e^x)

Examples of substitution:

1. $\int e^{\sqrt{x}}\,dx$

2. $\int \frac{3}{x - 2\sqrt{x}}\,dx$

3. $\int \frac{\cos(x)}{4 - \sin^2(x)}\,dx$

4. $\int e^x \cos(e^x)\sin^3(e^x)\,dx$

How would you *start* these?

1. $\int \tan^3(x)\sec(x)\,dx$

2. $\int x^2 \ln(x)\,dx$

3 $\int x\sqrt{5-x^2}dx$

4. $\int \frac{\sqrt{x^2-1}}{x^2}dx$

5. $\int \frac{x^2+1}{x^2-2x-3}dx$

6. $\int x\tan^{-1}(x)dx$

7. $\int \frac{dx}{\sqrt{4x^2+8x-12}}dx$

7.7 Approximating Integrals:

Despite our best efforts in 7.1-7.5**,** the vast majority of integrals can NOT be done with any of our methods.
So we usually have to approximate!

In this section we add two more approximation methods that are slightly more accurate. We already know left, right, and midpoint methods (but I included them below for completeness).

To approximate $\int_a^b f(x)dx$

1. Pick **n = number of subdivisions**.
   Compute $\Delta x = \frac{b-a}{n}$.
2. Label the tick marks: $x_i = a + i\Delta x$
3. Use an approximation method:

$$L_n = \Delta x[f(x_0) + f(x_1) + \cdots + f(x_{n-1})] \quad \text{(Left endpoint)}$$
$$R_n = \Delta x[f(x_1) + f(x_2) + \cdots + f(x_n)] \quad \text{(Right endpoint)}$$
$$M_n = \Delta x[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)] \quad \text{(Midpoint)}$$

New - Trapezoid Rule: (all the "middle terms" are multiplied by 2)

$$T_n = \frac{1}{2}\Delta x[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

New - Simpson's Rule: *n* must be even! (Alternating multiplying middle terms by 4 and 2)

$$S_n = \frac{1}{3}\Delta x[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Example: (Note: None of our methods can integrate this)
Estimate

$$\int_0^3 \sqrt{100 - x^3}\,dx$$

Here are how to use the left, right, midpoint and trapezoid rules with n = 3 subdivisions:

$$L_3 = (1)\left[\sqrt{100-(0)^3} + \sqrt{100-(1)^3} + \sqrt{100-(2)^3}\right] \approx 29.5415$$

$$R_3 = (1)\left[\sqrt{100-(1)^3} + \sqrt{100-(2)^3} + \sqrt{100-(3)^3}\right] \approx 28.0855$$

$$M_3 = (1)\left[\sqrt{100-(0.5)^3} + \sqrt{100-(1.5)^3} + \sqrt{100-(2.5)^3}\right] \approx 29.0091$$

$$T_3 = \frac{1}{2}(1)\left[\sqrt{100-(0)^3} + 2\sqrt{100-(1)^3} + 2\sqrt{100-(2)^3} + \sqrt{100-(3)^3}\right] \approx 28.8135$$

Here is how to use Simpson's rule with n = 6 subdivisions (n has to be even to use this method):

$$S_6 = \frac{1}{3}\cdot\frac{1}{2}\left[\sqrt{100-(0)^3} + 4\sqrt{100-(0.5)^3} + 2\sqrt{100-(1)^3} + 4\sqrt{100-(1.5)^3} + 2\sqrt{100-(2)^3} + 4\sqrt{100-(2.5)^3} + \sqrt{100-(3)^3}\right] \approx 28.9441$$

"Actual" Value (to 8 places after the decimal): 28.94418784